

On the dynamical emergence of de Sitter spacetime

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Abstract

We present and discuss an asynchronous coordinate system covering de Sitter spacetime, notably in a complete way in 1+1 dimensions. The new coordinates have several interesting cosmological properties: the worldlines of comoving ($x^i = \text{const}$) observers are geodesics, cosmic time is finite in the past, and the coordinates asymptotically tend to that of a flat Robertson & Walker model at large times. This analysis also provides an argument in favor of the natural emergence of an equation of state of the type $p = -\rho$ in the context of the standard cosmological model.

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1 Introduction

De Sitter spacetime [1, 2] plays a central role in cosmology. Its special status is justified on both historical and physical grounds. It was the first expanding world model ever proposed [3], the very one predicting the cosmological redshift of light, and the very one, together with Minkowski and anti-de Sitter spacetimes, to embody the maximal degree of symmetry in both space and time. Despite its idealistic nature – this spacetime cannot accommodate matter – the de Sitter model is still nowadays an essential framework to understand the critical properties of the primordial inflationary universe as well as the asymptotic future state of a universe dominated by the cosmological constant. Indeed, during both these epochs, the evolution of the universe can be effectively modelled as a de Sitter stage with slightly broken time-translational symmetries.

There are many equivalent geometrical definitions of a de Sitter spacetime, each of them illustrating and emphasising complementary facets of its metric. It can be defined, as the spacetime admitting the de Sitter group $O(4, 1)$ as group of symmetries [4], *i.e.*, the 4-dimensional spacetime that results from embedding a hyperboloid in a 5-dimensional Minkowski spacetime. It can also be obtained via the analytical continuation of a metric describing a positive curvature space with Euclidean signature into a constant curvature space of Lorentzian signature [5]. In a cosmological context, the dynamical emergence of the de Sitter spacetime as the vacuum solution to the Einstein's Field Equations (EFE) with a positive cosmological constant [1, 2] is of more interest. In other terms, the existence of a cosmological component with an effective energy density ρ and pressure p related by an equation of state $p = -\rho$ is mandatory if we are to obtain the de Sitter spacetime as solution of the EFE.

A large variety of coordinate systems is known for de Sitter space (see, e.g., [6–8]). In the context of General Relativity, because of local diffeomorphism invariance, coordinate systems play no role in the formulation of fundamental physical laws, only diffeomorphism invariants matter. On the other hand, a poor choice of local coordinates can sometimes obscure the interpretation of physical phenomena and the fundamental properties of spacetime. As an example, suitably tailored coordinate systems, which embody specific symmetries of a physical system, may help to solve differential equations [9]. Furthermore, coordinate systems that cover larger patches of spacetime are specifically useful when dealing with physical phenomena that are non-localized [10] or which are sensitive to global properties of spacetime [11].

The purpose of this paper is to discuss the cosmological properties of a specific comoving geodesic coordinate system, describing a 3D space with translational and rotational symmetries, which admits the flat Robertson & Walker (R&W) coordinate system as future asymptotic attractor. Moreover the only cosmological solution of the EFE (without the cosmological term) corresponds to a perfect fluid component with an effective equation of state $p = -\rho$. In other terms, more than a mere option in the space of the possible outcomes of cosmological experiments, such an ‘exotic’ equation of state naturally emerges as a necessary ingredient of the universe for a specific class of comoving geodesic observers which becomes R&W at later epochs.

Throughout this paper we suppose that cosmic spacetime is a Riemannian manifold (\mathcal{M}, g) and we adopt the sign convention $(+, -, -, -)$ for the metric g . If not indicated otherwise, we adopt Einstein's convention summing over repeated indices. Greek indices run over spacetime coordinates (from 0 to 3) and latin indices run over space coordinates (from 1 to 3). In particular, $x^i = (x, y, z)$ indicate the standard cartesian coordinates. We set the light speed $c = 1$ but we keep the gravitational constant G explicit in the equations.

2 A new class of comoving observers

In this section we introduce the cosmological coordinate system x^μ that is central to our analysis. Specifically, we show that one can define a class of comoving observers ($x^i = \text{const}$), freely falling in

the cosmological gravitational field, so that their hypersurfaces of constant cosmic time are maximally symmetric. We also discuss how this class of cosmological observers relates to the standard R&W ones.

It is a compelling philosophical argument, and, at the same time, a well established observational fact that the universe looks uniform to any R&W observer, *i.e.*, to observers that are freely falling in the cosmic gravitational field (geodesic motion) and that are at rest (comoving) with the surrounding cosmic fluid [12–16]. It is worth noticing that, as shown in appendix A, the most general class of geodesic and comoving cosmological observers (t, x^i) for which the hypersurfaces of constant time are flat and maximally symmetric can be described via the infinitesimal line element

$$ds^2 = dt^2 + 2q_i dt dx^i - a(t)^2 \delta_{ij} dx^i dx^j, \quad (1)$$

where q_i are constants, $a(t)$ is an arbitrary function of the cosmic time t , and δ_{ij} is the Kronecker symbol. As a matter of fact, this metric has (at least) six spatial isometries, three translations \mathbf{T}_i and three rotations \mathbf{R}_i , represented by the infinitesimal Killing vectors

$$\mathbf{T}_i = \frac{\partial}{\partial x^i}, \quad \mathbf{R}_i = \epsilon_{ij}^{k} \left(x^j - q^j \int \frac{dt}{a(t)^2} \right) \frac{\partial}{\partial x^k} \quad \text{for } i = 1, 2, 3 \quad (2)$$

where $q^i = \delta^{ij} q_j$ and where ϵ_{ij}^{k} is the Levi-Civita symbol¹.

In the local, comoving, geodesic coordinate system x^μ , cosmic time is tilted, it is non-orthogonal to spatial hypersurfaces. For such observers, indeed, the time of flight of photons along null geodesics is direction dependent. Consider a light cone with vertex $A \in \mathcal{M}$ and a light signal over some infinitesimal path. The signal must satisfy $ds^2 = 0$, and thus, by assuming for the sake of clarity that only $q_3 \equiv q \neq 0$,

$$dt_{\pm} = -q dz \pm \sqrt{q^2 dz^2 + a^2(dx^2 + dy^2 + dz^2)}, \quad (3)$$

where the plus sign corresponds to the future light cone at A , the negative sign to the past light cone. There are two solutions for dt , corresponding to the two ways of taking the path; and both solutions are time dependent, therefore the metric is not static [17]. Note that the asymmetry in the propagation of light, *i.e.*, the time irreversibility, is induced by the preferred direction in time (and not in space) brought about by the time-space cross term q . It is thus worth mentioning, to avoid any possible misunderstanding, that clock synchronisation between the fundamental observers, is still possible in the frame $q \neq 0$, and it can be operationally carried out, at least in principle, via light exchanges (e.g. [18]).²

It is worth clarifying here the relation between the ‘tilted-time’ cosmological observers and the standard R&W ones (τ, y^i) , those characterised by the line element

$$ds^2 = d\tau^2 - a(\tau)^2 \delta_{ij} dy^i dy^j, \quad (4)$$

¹ ϵ_{ij}^{k} equal 1 if (i, j, k) is an even permutation of $(1, 2, 3)$, -1 if (i, j, k) is an odd permutation of $(1, 2, 3)$ and 0 for repeated indices.

²Time asymmetry is exactly what we experience in the universe under real ‘observing conditions’. For example, consider a flat R&W metric $ds^2 = d\tau^2 - a^2(dx^2 + dy^2)$, and look at the physical distance as a new spatial coordinate

$$\begin{aligned} \bar{\tau} &= \tau \\ \bar{x} &= a x \\ \bar{y} &= a y \end{aligned}$$

then, in the “physical” reference frame, the metric element reads

$$ds^2 = [1 - \mathcal{H}^2(\bar{x}^2 + \bar{y}^2)] d\bar{\tau}^2 + 2\mathcal{H}(d\bar{x} + d\bar{y})d\bar{\tau} - (d\bar{x}^2 + d\bar{y}^2),$$

where \mathcal{H} is the Hubble parameter of the R&W observers.

where τ is the cosmic time of standard cosmology. Given that the line elements (1) and (4) are isometric, there is a diffeomorphism that maps the coordinates x^α into the coordinates y^α . To this purpose we consider a class of freely falling particles that, at present time τ_0 , move with a given velocity v_0 with respect to the R&W observers. Without lack of generality, we can orient both the $x^3 \equiv z$ and y^3 axes in such a way that they are collinear to the direction of this velocity. The geodesic motion of these particles (see Eqs. (82) in Appendix B) is given, in terms of the R&W y^i coordinates, by

$$y^1(\tau) = x, \quad (5)$$

$$y^2(\tau) = y, \quad (6)$$

$$y^3(\tau) = z \pm \int_{\tau_0}^{\tau} \frac{dt}{a(t)\sqrt{1+q^{-2}a^2(t)}}, \quad (7)$$

where

$$q \equiv \sqrt{\frac{v_0^2}{1-v_0^2}} \quad (8)$$

and where the constant values x, y, z define the cosmological observers of the standard cosmological model, *i. e.*, the geodesic observers with null velocity ($q = v_0 = 0$) at present time. We now show that the comoving observers $x^i = \text{const}$ in the tilted time coordinate system are those in motion with constant velocity v_0 with respect to R&W observers, those that co-moves with the freely falling boosted particles. Indeed, by setting $z = y^3(\tau)$ and by taking the differential, we obtain that the infinitesimal coordinate transformation

$$dz = dz + \frac{q}{a^2} dt \quad (9)$$

$$d\tau = dt \sqrt{1 + \frac{q^2}{a^2}} \quad (10)$$

maps (4) into the infinitesimal line element

$$ds^2 = dt^2 + 2qdt dz - a(t)^2(dx^2 + dy^2 + dz^2), \quad q = \text{const}. \quad (11)$$

Since we can always orient the coordinate system x^μ such that two of the q_i are made zero, from now on, and without lack of generality, we will set $q_1 = q_2 = 0$ and $q_3 = q$.

The first thing worth noticing is that, as anticipated, and now made explicit by Eq. (8), the class of boosted comoving observers contains the R&W one in the limiting case $v_0 = 0$. More interestingly, equation (10) shows that the hypersurfaces of constant time for the boosted observers are the very same hypersurfaces of the R&W observer. This explains why spatial hypersurfaces are maximally symmetric, in other terms why the cosmological principle holds also for this class of observers. The cosmological time of boosted observers, instead, ticks at a different, slower, rate with respect to the cosmic time of the R&W observers. We immediately deduce that cosmic time *is not univocally defined* simply by requiring the clocks to be freely falling in the cosmological gravitational field. In other terms *one cannot transform the coordinates t into τ and require both to measure cosmic time*. An additional physical structure is required if we are to specify cosmic time in an unambiguous, universal way. It is true that the intrinsic geometry of the cosmological manifold is invariant, *i. e.*, a coordinate transformation does not change the isometries of the metric, but is equally fundamental to recognise that one cannot define two different cosmological observers (in the sense of Eq. (7)) and impose that they are both comoving with the same cosmological fluid. The observers (1) and (4), although related by a diffeomorphism, do not describe the same physical system owing to the fact that they are both required to have a null spatial velocity ($\mathbf{u}_t \propto \partial_t$ and $\mathbf{u}_\tau \propto \partial_\tau$ respectively) with respect to the *substratum*. One therefore needs to explore, using the EFE, which cosmic fluid,

if any, is uniformly comoving with the new reference frame. The matter content of the universe will then eventually define the geometry of the cosmic spacetime which is compatible with this new hypothetical class of observers.

To get a grasp on the different physical content of the world model (1) with respect to the standard one (cf. Eq. (4)), consider a time-like velocity field describing the motion of fluid elements $dx^\alpha/d\lambda = u^\alpha(x^\beta)$ (where λ is an affine parameter, where we assume that the world line of a fluid passes through every point x^β of a certain region in spacetime, and, also, that the velocity field of the fluid is differentiable in this whole region) and assume, further, that the velocity vector is normalised to unity

$$||\mathbf{u}|| = \sqrt{u^\mu u^\nu g_{\mu\nu}} = 1. \quad (12)$$

Its covariant derivative can be decomposed in terms of its trace (Θ), its trace-free symmetric part ($\sigma^\mu{}_\nu$), and its (trace-free) antisymmetric part ($\omega^\mu{}_\nu$) as

$$u^\mu{}_{;\nu} = \frac{1}{3}\Theta h^\mu{}_\nu + \sigma^\mu{}_\nu + \omega^\mu{}_\nu, \quad (13)$$

where $h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ is the projection tensor on hypersurfaces orthogonal to u^α . The three coordinate invariant scalar quantities defined as

$$\Theta = u^\alpha{}_{;\alpha} \quad (\text{expansion}), \quad (14)$$

$$\sigma = \sqrt{\sigma^\mu{}_\nu \sigma^\nu{}_\mu} \quad (\text{shear}), \quad (15)$$

$$\omega = \sqrt{\omega^\mu{}_\nu \omega^\nu{}_\mu} \quad (\text{vorticity}), \quad (16)$$

characterise, univocally, the evolution of the velocity field and a straightforward evaluation of these quantities shows that if the fluid is comoving ($\mathbf{u} = (1, 0, 0, 0)$) with the observers defined via Eq. (11) then

$$\Theta = 3\frac{\dot{a}}{a}, \quad (17)$$

$$\sigma = \sqrt{\frac{3}{2}}\frac{\dot{a}}{a}\frac{q^2}{a^2 + q^2}, \quad (18)$$

$$\omega = 0. \quad (19)$$

Notice that the shear scalar is different from zero only if the expansion scalar is different from zero and the time-space component q is also non-null. We note, incidentally, that by using the Raychaudhuri equation, the trace of the tidal tensor is

$$E[u^\mu]^\alpha{}_\alpha = \Theta_{;\alpha} u^\alpha + \frac{1}{3}\Theta^2 + \sigma^2 - \omega^2 \quad (20)$$

$$= -3\frac{\ddot{a}}{a} + \Delta\left(\frac{\ddot{a}}{a} - (1 - \Delta)\frac{\dot{a}^2}{a^2}\right), \quad (21)$$

where we define $\Delta \equiv q^2/(a^2 + q^2)$. In the special case where $\ddot{a}/a = (1 - \Delta)(\dot{a}/a)^2$, we recover the FLRW result $E(u^\mu)^\alpha{}_\alpha = -3\ddot{a}/a$.

The centrality of the assumption $q = 0$ in singling out the standard cosmological observers can now be understood in terms of a necessary and sufficient condition for an observer to be in the R&W class. A cosmological observer is of the R&W type if and only if it obeys EFE with a *perfect fluid* source that has zero vorticity, shear and acceleration [19].³ Equation (18) shows now that the

³Note that, an equivalent definition makes no reference at all to Einstein's field equations: an observer is R&W iff (1) the spacetime admits a foliation into spacelike hypersurfaces of constant curvature, (2) the congruence of the observers worldlines are orthogonal to the leaves of the foliation and shear-free geodesics, and (3) the expansion scalar of the geodesic congruence has its gradient tangent to the geodesics.

boosted observer implements these properties except for the shear-free condition. Although the new coordinates are defined via a diffeomorphism (cf. Eqs. (9) and (10)), the shear scalar is not equal to the value calculated using R&W coordinates. Indeed we have imposed that the velocity of the fluid is null in the boosted coordinates, *i.e.*, it is not fixed by the coordinate transformation itself. Note, however, that the R&W value can be recovered in the asymptotic limit in which $a(t) \gg q$. In other terms it is the dynamical evolution of the scale factor that eventually drives the general metric (75) into the R&W special limit. That this effectively happens, at large cosmic times, will be seen in §3.2.

In what follows we shall be investigating the physical consequences of relaxing the assumption that the cosmic time measured by a class of comoving observers is orthogonal to space-like hypersurfaces. There are several reasons that make it natural to explore the aftermath of abandoning this assumption. The simplest is that, as we will see in the next sections, this class of comoving observers has an interesting level of generality, encompassing the R&W subclass in a given well-defined limit. The most compelling, however, is that any increase in generality which can be brought about without introducing new hypotheses, but only by removing previous restrictions, will be of advantage in shedding light on the intrinsic nature of the cosmic spacetime.

3 Dynamics

In this section we explore which cosmic component, if any, satisfies the cosmological principle in the reference frame of the boosted comoving geodesic observers. We do this by computing the EFE for the metric (11) and by assuming a general relativistic imperfect fluid stress energy tensor.

3.1 Einstein's Field Equations

A vigorous research program is nowadays conducted to assess whether the predictions of the EFE can be safely extrapolated on cosmological scales of the order of the Gigaparsec [20–27]. Current evidences seem to suggest that this is indeed the case only if the EFE is complemented by an additional term, the cosmological constant Λ [28–33]. Since our goal is to highlight the dynamical emergence of cosmic components that behave as an effective cosmological constant (in the sense that their equation of state is $p = -\rho$), we will assume, right from the beginning, that the EFE contain no additional Λ term. The Einstein tensor is

$$G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}R^\alpha{}_\alpha\delta^\mu{}_\nu \quad (22)$$

and it has the following non-vanishing components

$$G^0{}_0 = 3(1 - \Delta)H^2, \quad (23)$$

$$G^3{}_0 = -2\frac{q}{a^2}(1 - \Delta)(\dot{H} + \Delta H^2), \quad (24)$$

$$G^i{}_i = (1 - \Delta)(2\dot{H} + (3 + 2\Delta)H^2) \quad (\text{no summation on repeated indices}). \quad (25)$$

Note that $G^0{}_i = 0$, and in particular $G^0{}_3 = 0$. Following [34], the stress-energy tensor of a general relativistic imperfect fluid can be modelled as

$$T_{\mu\nu} = \rho(t)u_\mu u_\nu - p(t)h_{\mu\nu} - \eta(t)h^\alpha{}_\mu h^\beta{}_\nu \sigma_{\alpha\beta} - \zeta(t)\Theta h_{\mu\nu} - \chi(t)(h^\alpha{}_\mu u_\nu + h^\alpha{}_\nu u_\mu)(T_{;\alpha} + T\dot{u}_\alpha) \quad (26)$$

where $h_{\mu\nu}$ is the projection tensor, $\rho = \rho(t)$ and $p = p(t)$ are respectively energy density and isotropic pressure of the fluid, and $\eta(t)$, $\zeta(t)$ and $\chi(t)$ can be interpreted as *shear viscosity*, *bulk viscosity* and *heat conduction*. The first line contains perfect fluid terms, while the dissipative terms are in the second line. Due to homogeneity and isotropy, we allow all these 5 functions to depend only on time. This should hold for the temperature T defined such that the energy density $\rho(T, n)$ is equal

to the comoving energy density $T_{\alpha\beta}u^\alpha u^\beta$ at thermal equilibrium. Writing explicitly the temperature as $T = T(t)$ induces the tensor multiplying $\chi(t)$ to vanish. If we additionally impose that the fluid is comoving ($\mathbf{u} = (1, 0, 0, 0)$) with the tilted-time observers x^μ , we are left with the following components

$$\begin{aligned} T^0_0 &= \rho(t) \\ T^0_3 &= q(\rho(t) + p(t) - \frac{2}{3}\Delta H\eta(t) + 3(1 - \frac{1}{3}\Delta)H\zeta(t)) \\ T^1_1 &= T^2_2 = -p(t) - \frac{1}{3}\Delta H\eta(t) - (3 - \Delta)H\zeta(t) \\ T^3_3 &= -p(t) - \frac{2}{3}\Delta H\eta(t) - (3 - 5\Delta)H\zeta(t). \end{aligned} \quad (27)$$

Note, incidentally, that $T^3_0 = 0$. By comparing the above equations with (25), we see that the components G^i_i are the same whereas T^i_i are not. By enforcing the equality of the components T^i_i , a stringent constraint between shear and bulk viscosity

$$\eta(t) = 12\zeta(t) \quad (28)$$

results. As a consequence, we obtain the following four *Friedmann-like* equations

$$3H^2(1 - \Delta) = 8\pi G\rho, \quad (29)$$

$$-2\frac{q}{a^2}(1 - \Delta)(\dot{H} + \Delta H^2) = 0, \quad (30)$$

$$0 = 8\pi Gq(\rho + p + 3(1 - 3\Delta)H\zeta), \quad (31)$$

$$(1 - \Delta)(2\dot{H} + (3 + 2\Delta)H^2) = -8\pi G(p + 3(1 + \Delta)H\zeta). \quad (32)$$

Moreover, the conservation equation, $T^\mu_{\nu;\mu} = 0$, yields the additional, non-independent, equations (for $q \neq 0$)

$$\dot{\rho} = -(3 - \Delta)H[\rho + p + 3(1 + \Delta)H\zeta] \quad (33)$$

$$\dot{p} = 6\Delta(1 + 3\Delta)H^2\zeta - 3(1 - 3\Delta)(\dot{H}\zeta + H\dot{\zeta}). \quad (34)$$

We are thus led to the conclusion that if viscosity is zero then the pressure (if any) has to be constant in time.

3.2 Solutions

If we set $q = 0$ ($\Delta = 0$) in equation (32) we obtain the usual Friedmann equations for an imperfect fluid:

$$3H^2 = 8\pi G\rho \quad (35)$$

$$2\dot{H} + 3H^2 = -8\pi G(p + 3H\zeta). \quad (36)$$

Notice that for the case of the Friedmann equations, the equation of state parameter $w = p/\rho$ is completely unconstrained. If $q \neq 0$ ($\Delta \neq 0$), instead, we obtain

$$3H^2(1 - \Delta) = 8\pi G\rho, \quad (37)$$

$$\dot{H} + \Delta H^2 = 0, \quad (38)$$

$$p = -\rho, \quad (39)$$

$$\zeta = 0. \quad (40)$$

that is, no real fluids, only perfect ones, are solutions of the EFE. Additionally, their equation of state is constrained and violates the strong energy condition, *i.e.*, the requirement $\rho + 3p \geq 0$.

By inserting eqs. (39) and (40) into (33) we deduce that the energy density ρ is constant in time. The general solution of equation (37) is thus

$$a(t) = e^{\mathcal{H}_\Lambda(t-t_0)} - \frac{q^2}{4} e^{-\mathcal{H}_\Lambda(t-t_0)}, \quad (41)$$

where t_0 is an arbitrary integration constant and where we have expressed the constant density component as the vacuum energy seen by a R&W observer

$$\rho = \text{const} \equiv \frac{3\mathcal{H}_\Lambda^2}{8\pi G}. \quad (42)$$

Indeed, by definition, any freely falling observer sees the same vacuum (which is Lorentz invariant). Note that, unlike H , the Hubble parameter \mathcal{H}_Λ for a R&W observer is constant.

If $q \neq 0$, then, in the finite past, the geodesic worldlines intersect at a point, *i. e.*, the scale factor vanishes. By imposing that this happens for $t = 0$ we can fix the arbitrary integration constant

$$t_0 = -\frac{1}{\mathcal{H}_\Lambda} \ln \frac{q}{2}, \quad (43)$$

and we can recast the solution (41) in the form

$$a(t) = q \sinh(\mathcal{H}_\Lambda t). \quad (44)$$

Note that even though H is not a constant, we are dealing with a de Sitter spacetime. In fact, evaluating the Ricci scalar, we find

$$R^\alpha_\alpha = -6(1 - \Delta)(\dot{H} + (2 + \Delta)H^2) = -12(1 - \Delta)H^2 = -32\pi G\rho, \quad (45)$$

which is manifestly constant. That this solution represents a 4D de Sitter universe is also made evident by the fact that the spacetime has maximal symmetry, that is 10 Killing vectors. Indeed by solving the Killing equations ((74) in Appendix A) using the metric (1) and the scale factor (44) we find, on top of the spatial translations \mathbf{T}_i and rotations \mathbf{R}_i (see cf. Eq. (2)), also the time translation isometry \mathbf{T}_0 and three Lorentz isometries (boosts) \mathbf{B}_i . Explicitly,

$$\mathbf{T}_0 = \mathcal{H}_\Lambda^{-1} \tanh(\mathcal{H}_\Lambda t) \partial_t - (x^k + q^k (q^j q_j)^{-2} \mathcal{H}_\Lambda^{-1} \tanh(\mathcal{H}_\Lambda t)) \partial_k, \quad (46)$$

$$\mathbf{T}_i = \partial_i, \quad (47)$$

$$\begin{aligned} \mathbf{B}_i = & \mathcal{H}_\Lambda^{-1} (x_i \tanh(\mathcal{H}_\Lambda t) + q_i (q^j q_j)^{-2} \mathcal{H}_\Lambda^{-1}) \partial_t \\ & - (x_i + q_i (q^j q_j)^{-2} \mathcal{H}_\Lambda^{-1} \coth(\mathcal{H}_\Lambda t)) (x^k + q^k (q^j q_j)^{-2} \mathcal{H}_\Lambda^{-1} \tanh(\mathcal{H}_\Lambda t)) \partial_k \\ & + \frac{1}{2} \left[\mathcal{H}_\Lambda^{-2} (q^j q_j)^{-2} + x_k (x^k + 2q^k (q^j q_j)^{-2} \mathcal{H}_\Lambda^{-1} \tanh(\mathcal{H}_\Lambda t)) \right] \partial_i, \end{aligned} \quad (48)$$

$$\mathbf{R}_i = \epsilon_{ij}^{k} (x^j + q^j (q^l q_l)^{-2} \mathcal{H}_\Lambda^{-1} \coth(\mathcal{H}_\Lambda t)) \partial_k, \quad (49)$$

where the Kronecker symbol δ_{ij} is used to rise/lower spatial indexes. These Killing vectors obey the non vanishing de Sitter commutation relations

$$[\mathbf{T}_0, \mathbf{T}_i] = \mathbf{T}_i, \quad [\mathbf{T}_0, \mathbf{B}_i] = -\mathbf{B}_i, \quad [\mathbf{T}_i, \mathbf{B}_j] = \delta_{ij} \mathbf{T}_0 + \epsilon_{ij}^{k} \mathbf{R}_k \quad (50)$$

$$[\mathbf{T}_i, \mathbf{R}_j] = -\epsilon_{ij}^{k} \mathbf{T}_k, \quad [\mathbf{B}_i, \mathbf{R}_j] = -\epsilon_{ij}^{k} \mathbf{B}_k, \quad [\mathbf{R}_i, \mathbf{R}_j] = -\epsilon_{ij}^{k} \mathbf{R}_k \quad (51)$$

i. e., the commutations of the $O(4, 1)$ group in the basis defined by our coordinates.

In summary, if we describe gravity using the EFE without the cosmological constant, the only model of the universe that looks uniform to freely falling, comoving, boosted observers is de Sitter. In other terms, the universe cannot contain a geodesic fluid, comoving with the observers, other than

a perfect one with an equation of state parameter $w = p/\rho = -1$. This analysis complements the standard dynamical derivation of de Sitter spacetime as the vacuum solution of the EFE *augmented* by the cosmological constant in a universe which looks uniform to R&W observers. It highlights that a component with an effective equation of state $p = -\rho$ (be it a cosmological constant or dark energy) is not only a mathematical option, it is, instead, and under very general conditions, a necessary condition imposed by requiring the validity of the cosmological principle for a general class of freely falling, comoving, observers. In other terms, such an ‘exotic’ equation of state naturally emerges as an essential and not ancillary ingredient of the standard model of the universe if space is to show translational and rotational symmetries.

Besides offering new insights into the dynamical emergence of de Sitter space times in cosmology, this result also highlights, in some sense, the intrinsic non-Machian nature of the general relativistic theory of gravity. The issue whether general relativity realises Machian ideas has always been a controversial one, especially because, in spite of much discussions and debates, it has never been entirely clear what the Mach principle is. With this caveat in mind, it is nonetheless interesting to note that while the theory allows for the possibility of the boosted motion of observers within the gravitational field generated by a uniform distribution of matter, a global boost of a uniform distribution of matter cannot generate that very same gravitational field. In other terms, from this arguments it seems that there is no such principle as the relativity of inertia embedded in the EFE.

3.3 Slicing the de Sitter spacetime

Equation (41) shows that the usual de Sitter expansion law (in flat R&W coordinates) is recovered in the special case $q = 0$. More interestingly, also at large times the scale factor expands as in the flat R&W coordinates. Indeed, as soon as $t \gg \mathcal{H}_\Lambda^{-1}$, q becomes subdominant with respect to $a(t)$, the shear scalar vanishes (cf. Eq. (18)), and the boosted geodesic observer (11) asymptotically converges to the flat R&W one, *i.e.*, the boosted and the R&W frames are essentially undistinguishable.

The coordinate transformation that maps the comoving coordinates of the boosted observer (t, x, y, z) into the flat R&W coordinates of a de Sitter space time are obtained by embedding a 4D hyperboloid in a 5D Minkowskian embedding space $(Y_0, Y_1, Y_2, Y_3, Y_4)$ with coordinate transformation

$$Y_0 = \mathcal{H}_\Lambda^{-1} \left[\left(1 + \frac{1}{2}(x^2 + y^2 + z^2) \right) \sinh(\gamma t) + z \cosh(k t) \right], \quad (52)$$

$$Y_1 = -\mathcal{H}_\Lambda^{-1} \left[z \cosh(\gamma t) + \frac{1}{2}(x^2 + y^2 + z^2) \sinh(\gamma t) \right], \quad (53)$$

$$Y_2 = \mathcal{H}_\Lambda^{-1} [\cosh(\gamma t) + z \sinh(\gamma t)], \quad (54)$$

$$Y_3 = \mathcal{H}_\Lambda^{-1} x \sinh(\gamma t), \quad (55)$$

$$Y_4 = \mathcal{H}_\Lambda^{-1} y \sinh(\gamma t), \quad (56)$$

where γ is an arbitrary parameter. One can verify that these coordinates define a 4-dimensional de Sitter hyperboloid,

$$Y_0^2 - Y_1^2 - Y_2^2 - Y_3^2 - Y_4^2 = -\mathcal{H}_\Lambda^{-2}, \quad (57)$$

where $Y_1 = Y_2 = Y_3 = Y_4 = 0$ defines its axis of symmetry and \mathcal{H}_Λ^{-1} is its radius at $Y_0 = 0$. Also, one can verify that the coordinate transformation (52) to (56) induces the tilted time metric (11) on the hyperboloid,

$$\begin{aligned} ds^2 &= dY_0^2 - dY_1^2 - dY_2^2 - dY_3^2 - dY_4^2 \\ &= dt^2 + 2q dt dz - q^2 \sinh^2(\mathcal{H}_\Lambda t) (dx^2 + dy^2 + dz^2), \end{aligned} \quad (58)$$

once one identifies

$$q = \mathcal{H}_\Lambda^{-1} = \gamma^{-1}. \quad (59)$$

This constraint between the constant vacuum energy density (parameterised by \mathcal{H}_Λ) and the velocity of the boosted observer (parameterised by q) follows from fixing the gauge freedom in (41), *i.e.*, by defining the zero point of cosmic time as the specific moment at which the scale factor goes to zero.

A visual picture of the de Sitter space foliation implied by our geodesic comoving coordinates can be obtained by setting $x = y = 0$ in (52)-(56). The resulting 2D de Sitter hyperboloid, together with the coordinate grid, is shown in fig. 1. Noteworthily, the tilted time coordinate system covers all of the hyperboloid, contrary to the flat R&W coordinates which cover only half of it. As expected, when the time constraint $t \geq 0$ generated by the EFE is considered, the portion of de Sitter space covered by the two coordinate systems is identical. Finally note that the coverage is not complete in 4D (only 85% is charted).

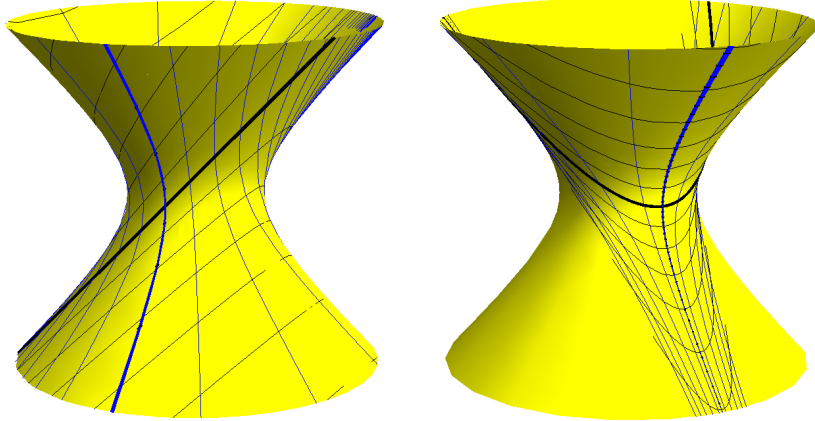


Figure 1: *Left*: foliation of de Sitter spacetime with the tilted time coordinates. Blue lines represent comoving geodesics $z = \text{const}$, black lines represent hypersurfaces of constant cosmic time t . These coordinates cover the entire hyperboloid for $-\infty < t < \infty$ and $-\infty < z < \infty$. The thicker lines indicate the curves $z = 0$ and $t = 0$. *Right*: de Sitter foliation in the standard flat R&W coordinates for $-\infty < \tau < \infty$ and $-\infty < z < \infty$.

4 Origin of time in de Sitter spacetime

Although the scale factor $a(t)$ vanishes, there is no physical singularity at $t = 0$, in the sense that the curvature invariant (45) is well defined. Instead the shear scalar (cf. Eq. (18))

$$\sigma(t) = \sqrt{\frac{8}{3}} \frac{\mathcal{H}_\Lambda}{\sinh(2\mathcal{H}_\Lambda t)} \quad (60)$$

diverges when $t \rightarrow 0$. By exploiting the time dilation equation (9), we obtain the explicit relation between the cosmic time measured by boosted and R&W observers

$$t = \mathcal{H}_\Lambda^{-1} \left(e^{\mathcal{H}_\Lambda \tau} + \sqrt{1 + e^{2\mathcal{H}_\Lambda \tau}} \right), \quad (61)$$

which makes manifest the mapping of the R&W epoch $\tau = -\infty$ into the coordinate $t = 0$ of the tilted time coordinate system. Note also that the two clocks tick at the same pace at large cosmic times, as soon as $t \gg \mathcal{H}_\Lambda^{-1}$.

We further discuss this specific remapping of the cosmic time by calculating the time evolution of a ghost condensate [35] in the boosted frame. The ghost condensate is a hypothetical fluid that might fill the universe and that has the same effective equation of state of the cosmological constant ($\rho = -p$).

It can hence drive de Sitter expansion of the universe. However, unlike a cosmological constant, it is a physical fluid which is unstable under scalar perturbations. We consider its Lagrangian density $\mathcal{L} = -P$ where $P = -X + \frac{X^2}{M^4}$ and where $X = (1/2)g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$. By considering the kinetic term $X = M^4/2$, the equation of state of the effective perfect fluid associated to this cosmological component is $p = -\rho$, while the time evolution of the field is

$$\phi(t) = \phi(t_i) \pm \frac{M^2}{\mathcal{H}_\Lambda} \ln \left[\frac{\sinh(\mathcal{H}_\Lambda t)}{\sinh(\mathcal{H}_\Lambda t_i)} \right]. \quad (62)$$

Note that the initial time t_i cannot be set arbitrarily, but must satisfy the constraint $t_i \geq 0$. Also note that the ϕ -field evolves as in the standard flat R&W coordinates ($\phi = \phi_i + M^2(\tau - \tau_i)$) as soon as $\tau \sim t \gg \mathcal{H}_\Lambda^{-1}$.

5 Discussion and conclusions

An a-synchronous coordinate system is considered in which the fundamental observer of the universe is not anymore the R&W one – the freely falling observer that is comoving with a perfect fluid having zero shear, vorticity and acceleration – but a new one, that can still define a cosmic time and spatial hypersurfaces of constant curvature. Indeed, the coordinate system

$$ds^2 = dt^2 + 2q_i dt dx^i - a(t)^2 \delta_{ij} dx^i dx^j \quad (63)$$

obtained by relaxing the shear-free assumption and by considering as fundamental cosmological observers test particles that are boosted with a given constant velocity with respect to the R&W ones, is not, in principle, in conflict with the requirements of the cosmological principle. Specifically, a class of freely falling comoving ($x^i = \text{const}$) observers, with proper time not orthogonal to the flat spatial hypersurfaces, is identified for which the hypersurfaces of constant cosmic time are maximally symmetric, *i. e.* six independent isometries of the metric exist, three translations and three rotations.

We find that the only fluid that can be naturally accommodated in such a universe, if the cosmological principle must hold, is a perfect one with an effective equation of state $p = -\rho$. In particular, when the Einstein's field equations without Λ are considered, we find that the second time derivative of the scale factor of the metric is positive and equal to

$$\ddot{a} = (8\pi G/3)a \quad (64)$$

that is, the metric expands at the same accelerated pace as R&W tests particles in the cosmological model obtained by solving the EFE, augmented by Λ , in the vacuum. We also demonstrate that the resulting 1+3 spacetime is de Sitter, thus it possesses maximal symmetries. This analysis contributes to shedding light on the dynamical emergence of de Sitter space times in physics, and more specifically, provides an argument in favor of a natural emergence of the equation of state $w = -1$ in the context of the standard cosmological model. Indeed, despite its effects being dynamical, such as accelerating the expansion of a set of geodesic comoving observers, the very nature of this peculiar equation of state, appears to be intrinsically geometric, *i.e.*, associated to the requirement that space be maximally symmetric to a general class of cosmological observers.

We discuss several interesting cosmological properties of the tilted time coordinate system. First we show that it covers de Sitter spacetimes in a complete way in 1+1 dimensions, and we demonstrate that the coordinate time, which also plays the role of cosmic time, is finite in the past, a concept that we illustrate by showing that the time evolution of a specific dynamical scalar field with equation of state $p = -\rho$ – the ghost condensate – is bounded from below. Finally, at large times, when $a(t) \gg \sqrt{q_i q^i}$ the geodesic slicing of the de Sitter spacetime implied by this coordinate system asymptotically converges to the de Sitter slicing in standard flat R&W coordinates. That is, the flat

R&W observer is an attractor in the space of the general comoving observers specified by the metric element (1), irrespectively of the initial value of the constant parameters q_i .

As a byproduct, we discuss how this novel way of looking at de Sitter spacetimes also sheds light on the anti-Machian character of the Einstein's theory of gravity. The fact that it is not possible to generate a consistent gravitational field by boosting the uniform mass distribution of the universe, is at variance with the somewhat common interpretation of the Mach principle as a statement that inertia is a relative concept, *i.e.*, that boosting the sources of the gravitational field or the observers, generates physically equivalent inertial effects.

A way to generalise and improve the results presented in this paper is to figure out if there are comoving cosmological observers such that their proper time is non orthogonal to curved spatial hypersurfaces of maximal symmetry. An important aspect to explore is also the perturbative instability of such a metric. At a more speculative level, one might examine whether such sheared fluid verifies the conditions for generating a successful inflationary phase of the universe.

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5.1 Appendix A. The tilted time cosmological line element

Among the general class of reference frames in which the universe can be described and interpreted, one can single out as the fundamental ones those associated to test particles that are freely falling in the cosmic gravitational field, that is in geodesic motion $\mathbf{u}' = u'^\mu \frac{\partial}{\partial x^\mu} = 0$, where

$$\mathbf{u} = \frac{dx^\mu}{d\lambda} \frac{\partial}{\partial x^\mu} = u^\mu \frac{\partial}{\partial x^\mu} \quad (65)$$

is the velocity vector, λ is some affine parameter, and where $' \equiv d/d\lambda$. We assume that these fundamental cosmological observers are further characterised by a null spatial velocity $\mathbf{u} \propto \partial_t$, *i.e.*, they are comoving with the coordinate system ($x^i = \text{const}$).

The (0,0) component of the metric is univocally determined by assuming that the coordinate time of freely falling observers coincides with the proper time, *i.e.*,

$$g_{00} = 1. \quad (66)$$

This same condition allows to constrain the functional dependence of the mixed components (0, i) of the metric. Given the geodesic equations of motion

$$u'^\mu = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0 \quad (67)$$

one obtains, for comoving observers, $\Gamma^\mu_{00} = 0$ or, equivalently,

$$\frac{\partial g_{0i}}{\partial t} = 0 \quad i = 1, 2, 3 \quad (68)$$

that is, the cross-terms between the time component and the space components can be at most generic functions of space positions.

Any non-zero 4-displacement with $dt = 0$ takes place in what is generally called a spatial hypersurface σ_t . Standard cosmology follows from the assumption that flat hypersurfaces have six isometries, three translations and three rotations, represented by the infinitesimal operators

$${}^{(3)}\mathbf{T}_i = \frac{\partial}{\partial x^i}, \quad {}^{(3)}\mathbf{R}_i = \epsilon_{ij}{}^k x^j \frac{\partial}{\partial x^k}, \quad i = 1, 2, 3. \quad (69)$$

These Killing vectors form a basis of the space of solutions of the Killing equation for the metric $\delta_{ij}dx^i dx^j$. The spatial isometries in a 4-dimensional spacetime with flat space sections will therefore be of the general type

$$\mathbf{T}_i = F_{\underline{i}}(t) \frac{\partial}{\partial x^i}, \quad \mathbf{R}_i = \epsilon_{ij}{}^k (G_{\underline{kj}}(t)x^j + H_{\underline{k}}^j(t)) \frac{\partial}{\partial x^k}, \quad i = 1, 2, 3 \quad (70)$$

where $F_{\underline{i}}(t)$, $G_{\underline{kj}}(t)$ and $H_{\underline{i}}^j(t)$ are, respectively, 3, 6 and 6 *a-priori* arbitrary functions of time and where the summation convention is disabled on underscored indices.

Since equations (70) obey the Euclidian commutation relations

$$[\mathbf{T}_i, \mathbf{T}_j] = 0, \quad (71)$$

$$[\mathbf{T}_i, \mathbf{R}_j] = -\epsilon_{ij}{}^k \mathbf{T}_k, \quad (72)$$

$$[\mathbf{R}_i, \mathbf{R}_j] = -\epsilon_{ij}{}^k \mathbf{R}_k, \quad (73)$$

Eq. (71) is trivially satisfied. Equation (72), instead, imposes $F_{\underline{i}}(t)$ to be an isotropic function, $F(t)$, while from equation (73) we deduce $G_{\underline{kj}}(t) = 1$ and $H_{\underline{i}}^j(t) = H^j(t)$.

We can further specify the components of the metric by requiring them to be solutions of the $6 \times 10 = 60$ Killing equations

$$\xi^\mu \frac{\partial g_{\alpha\beta}}{\partial x^\mu} + \frac{\partial \xi^\mu}{\partial x^\alpha} g_{\mu\beta} + \frac{\partial \xi^\mu}{\partial x^\beta} g_{\alpha\mu} = 0 \quad (74)$$

where $\xi^\mu(x) = \{(\mathbf{T}_1)^\mu, \dots, (\mathbf{R}_3)^\mu\}$. We solve these partial differential equations by making the ansatz

$$ds^2 = dt^2 + 2g_{0i}(x^j)dt dx^i - a(t)^2 (dx^2 + dy^2 + dz^2) \quad (75)$$

The translations \mathbf{T}_i yield the following non-trivial equations

$$2g_{0i}(x^j) \frac{dF}{dt} = 0, \quad (76)$$

$$-a(t)^2 \frac{dF}{dt} + F(t) \frac{dg_{0i}}{dx^j} = 0, \quad i = j \quad (77)$$

$$F(t) \frac{dg_{0i}}{dx^j} = 0, \quad i \neq j \quad (78)$$

where $i = 1, 2, 3$ and summation over indices is again omitted. Solving these equations we find that $F(t) = \text{const}$ and $g_{0i}(x^j) = \text{const}$. We can set without loss of generality $F(t) = 1$ and $g_{0i}(x^j) = q_i$. By solving the Killing equations for the rotations \mathbf{R}_i we obtain the equations

$$\delta^{ij} q_j + a(t)^2 \frac{dH^i}{dt} = 0, \quad i = 1, 2, 3 \quad (79)$$

and therefore

$$H^i(t) = - \int \frac{q^i}{a(t)^2} dt, \quad (80)$$

where we define $q^i = \delta^{ij} q_j$. This implies the following form for the Killing vectors

$$\mathbf{T}_i = \frac{\partial}{\partial x^i}, \quad \mathbf{R}_i = \epsilon_{ij}{}^k \left(x^j - q^j \int \frac{dt}{a(t)^2} \right) \frac{\partial}{\partial x^k} \quad \text{for } i = 1, 2, 3. \quad (81)$$

As a consequence, the general line element given by Eq. (1) verifies the cosmological principle.

6 Geodesics motion of freely falling observers

The geodesic equations (see Eq. (67)), in the R&W coordinate system (4), take the form

$$\frac{d^2\tau}{d\lambda^2} + a^2\mathcal{H} \left[\left(\frac{dy^1}{d\lambda} \right)^2 + \left(\frac{dy^2}{d\lambda} \right)^2 + \left(\frac{dy^3}{d\lambda} \right)^2 \right] = 0, \quad \frac{d^2y^i}{d\lambda^2} + 2\mathcal{H}\tau' \frac{dy^i}{d\lambda} = 0, \quad (82)$$

where $\mathcal{H} = 1/a(da/d\tau)$ is the Hubble parameter, $i = 1, 2, 3$ and λ is an affine parameter. The general solutions of this system are given in Eqs. (5)-(7). The equations of motion in the tilted time coordinate system (11) are instead given by

$$\frac{d^2t}{d\lambda^2} - q \frac{d^2z}{d\lambda^2} + a^2H \left[\left(\frac{dy^1}{d\lambda} \right)^2 + \left(\frac{dy^2}{d\lambda} \right)^2 + \left(\frac{dy^3}{d\lambda} \right)^2 \right] = 0, \quad \frac{d^2x}{d\lambda^2} + 2H\tau' \frac{dx}{d\lambda} = 0, \quad (83)$$

$$\frac{d^2y}{d\lambda^2} + 2H\tau' \frac{dy}{d\lambda} = 0, \quad \frac{d^2z}{d\lambda^2} + \frac{q}{a^2} \frac{d^2t}{d\lambda^2} + 2H\tau' \frac{dz}{d\lambda} = 0, \quad (84)$$

where $H = 1/a(da/dt)$ is the Hubble parameter. Incidentally, one may note that, as expected, the comoving observers $x^i = \text{const}$ are in geodesic motion.